

Near extremal Kerr entropy from AdS₂ quantum gravity

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Near extremal Kerr entropy from AdS₂ quantum gravity

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ABSTRACT: We analyze the asymptotic symmetries of near extremal Kerr black holes in four dimensions using the AdS₂/CFT₁ correspondence. We find a Virasoro algebra with central charge $c_R = 12J$ that is independent from the Virasoro algebra (with the same central charge) that acts on the degenerate ground state. The energy of the excitations is computed as well, and we can use Cardy's formula to determine the near extremal entropy. Our result is consistent with the Bekenstein-Hawking area law for near extremal Kerr black holes.

KEYWORDS: AdS-CFT Correspondence, 2D Gravity, Black Holes

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1 Introduction

The understanding of black hole entropy in string theory generally involves the detailed study of specific microscopic configurations. A general shortcoming of this approach is that it obscures the presumed universal physics underlying black hole thermodynamics. To address this situation there has recently been renewed interest in more generic black holes, including those that are of interest in the astrophysical setting. An important system for this research direction is the extreme Kerr black hole [1, 2]. The near horizon geometry of all extreme black holes has an AdS₂ component [3] and, for extreme Kerr, this feature

has been combined with diffeomorphism invariance to give what may be a more robust understanding of black hole thermodynamics.

In this paper we generalize these results further, by accounting for the entropy of near extreme Kerr black holes. This generalization is interesting because it introduces new theoretical features that are expected to be present also for generic black holes. Additionally, astrophysical applications often involve Kerr that are nearly extreme, but not fully extreme.

The conjectured description of extremal Kerr black hole is in terms of a chiral CFT with central charge [1]

$$c_L = 12J . \tag{1.1}$$

The internal structure of the black hole is encoded in the ground state degeneracy of this chiral CFT. In our description, the excitations responsible for the leading departure from extremality are described by the *same* CFT, except that we must focus on the *opposite* chirality. We compute the boundary current of this sector and derive the corresponding central charge which is

$$c_R = 12J . \tag{1.2}$$

The two chiral sectors thus have the same central charge. However, they should not be confused since they are quite different in other aspects.

Our result for the boundary charges determines the relation between excitation energy and CFT level h_R for the right movers. We find that, in the right sector, there is no ground state degeneracy in the classical limit. The entropy is carried by genuine excitations and counted by Cardy's formula

$$S_R = 2\pi\sqrt{\frac{c_R h_R}{6}} . \tag{1.3}$$

The result we find in the CFT agrees with the Bekenstein-Hawking area law for near-extreme Kerr black holes.

In the original Kerr/CFT [1], the near horizon Kerr geometry (NHEK) [4] is represented as a warped U(1) fibration over AdS₂. The L Virasoro algebra emerges from the diffeomorphisms acting non-trivially on the asymptotic space of the near horizon geometry, following [5–9]. In this construction only a U(1) subgroup of the Virasoro algebra commutes with the SL(2, ℝ) isometry of the AdS₂ geometry. Recently, a similar strategy was pursued to study the finite temperature effects for the Kerr black hole [10, 11].

The near extreme generalization we present is much closer in spirit to conventional AdS/CFT correspondence. We follow [12] (which in turn generalized [13]) and exploit a complementary set of diffeomorphisms which preserve the asymptotic AdS₂ geometry. This yields the R Virasoro algebra, with an SL(2, ℝ) subgroup that can be identified with the isometry group. Excitations to finite level of this Virasoro algebra gives the entropy above extremality.

Our implementation of the AdS/CFT correspondence is closely related to Sen's quantum entropy function [14–16]. For example, the boundary mass term we introduce to regulate the on-shell action is reminiscent of the Wilson line that one must introduce in the quantum entropy function. It would of course be interesting to make such relations

precise. However, our focus in this paper is to generalize the constructions to finite temperature. We anticipate that such a generalization applies to the quantum entropy function as well.

The interplay between the two chiral sectors of the near extreme Kerr/CFT can be illuminated by contemplating a more general theory describing black holes far from extremality, inspired by [17–20] and [21, 22].¹ Suppose such a theory has the structure of a string theory, i.e. two chiral sectors coupled by some level matching condition that operates on the zero-modes alone. One sector (R) has excitations with the ability to carry angular momentum, the other (L) has quanta that cannot carry angular momentum. The Kerr black hole rotates, so in this case a macroscopic number of R quanta with angular momentum are excited. In the extreme limit the R sector becomes a condensate and *only* the quanta that carry angular momentum are excited. In this case the classical entropy derives from the degeneracy of the L sector which, due to level matching, must be excited as well. The entropy of supersymmetric black holes similarly derives from excitations in the L sector but, unlike extremal Kerr, the R sector is in its supersymmetric ground state. The entropy of non-supersymmetric extreme black holes (like the $D0 - D6$ black holes) are the opposite, it comes entirely from excitations in the R sector, and so breaks supersymmetry completely.

According to this model, the near extreme Kerr black hole receive most of their entropy from the ground state entropy in the L sector, but non-extremality involves exciting the R sector beyond those quanta that carry the angular momentum. It is the R sector above the condensate that we probe in this paper.

We derive our results directly from the 2D perspective. However, it is also interesting to analyze $\text{AdS}_2/\text{CFT}_1$ correspondence by embedding into the more familiar $\text{AdS}_3/\text{CFT}_2$ correspondence [12, 23–28]. In this construction the states responsible for the ground state entropy are invariant under the $\text{SL}(2, \mathbb{R})$ that appears as an isometry. The excitations above the ground state are quite different: they have a dispersion relation quadratic in the energy, and they do transform under the $\text{SL}(2, \mathbb{R})$. It is the physics of those excitations we focus on.

This paper is organized as follows. In section 2 we present a near horizon limit of the Kerr black hole that maintains energy above extremality. A component of this near horizon region is asymptotically (and locally) AdS_2 . We also show how to present near extreme Kerr entropy as a sum of ground state entropy and excitations described by Cardy’s formula. In sections 3 through 6 we develop the properties of the near horizon Kerr from the perspective of AdS_2 quantum gravity. In section 3 we construct the effective 2D theory by dimensional reduction of the four dimensional solution and classify the solutions to this theory. In section 4 we renormalize the action by adding boundary counterterms, ensuring a well-defined variational principle and finite conserved charges. In section 5 we determine the diffeomorphisms that preserve the asymptotic AdS_2 behavior, as well as other gauge conditions. This will follow the procedure in [12] very closely, albeit with an effective 2D action that is slightly different from the one analyzed [12]. In section 6 we construct the

¹We describe this model a bit further in the final discussion.

conserved Noether charges associated to the asymptotic symmetries. We find that the charge associated to diffeomorphisms is a non trivial combination of the boundary stress tensor and the $U(1)$ current. In section 7 we employ our findings from previous sections to investigate the properties of the near extremal black hole. Comparison with the area law gives a perfect match. Finally, in section 8, we conclude by discussing some of the open problems and known clues that point to a more general CFT description of all 4D black holes, including the elusive Schwarzschild black holes.

2 The near extreme Kerr limit

In this section we isolate the near horizon geometry of the Kerr black hole in a manner that maintains some excitation energy above the extremal limit. We discuss the extreme and near extreme Kerr entropy.

2.1 Near horizon geometry

The general Kerr solution is given by

$$ds^2 = -\frac{\Sigma\Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} dt'^2 + \Sigma \left[\frac{dr^2}{\Delta} + d\theta^2 \right] + \frac{\sin^2 \theta}{\Sigma} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) \left[d\phi' - \frac{2a\mu r}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} dt' \right]^2, \quad (2.1)$$

with

$$\Delta = (r - r_-)(r - r_+), \quad r_{\pm} = \mu \pm \sqrt{\mu^2 - a^2}, \quad (2.2)$$

and

$$\Sigma = r^2 + a^2 \cos^2 \theta. \quad (2.3)$$

In our notation $\mu = G_4 M$ and $a = J/M$ are length scales, while J is dimensionless. The near horizon region is isolated by introducing the coordinates

$$r = \frac{1}{2}(r_+ + r_-) + \lambda U, \quad t' = \frac{t}{\lambda}, \quad \phi' = \phi + \frac{t}{\lambda(r_+ + r_-)}. \quad (2.4)$$

The dimensionless scaling parameter $\lambda \rightarrow 0$ in the near horizon limit, with t, U, θ, ϕ fixed.

For extremal black holes we take the near horizon scaling limit with $r_+ = r_-$ from the outset. The near horizon geometry becomes

$$ds^2 = \frac{1 + \cos^2 \theta}{2} \left[-\frac{U^2}{\ell^2} dt^2 + \frac{\ell^2}{U^2} dU^2 + \ell^2 d\theta^2 \right] + \ell^2 \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} \left(d\phi + \frac{U}{\ell^2} dt \right)^2, \quad (2.5)$$

where we defined

$$\ell^2 \equiv \frac{1}{2}(r_+ + r_-)^2 = 2\mu^2. \quad (2.6)$$

The metric (2.5) is known as Near Horizon of Extreme Kerr (NHEK) [4]. The components described by the (t, U) coordinates in the square bracket of (2.5) is an AdS_2 geometry, with radius ℓ given in (2.6).

The near horizon limit is easily modified to maintain some energy above extremality in the limit. We need to take the limit while tuning the black hole parameters such that the scale ϵ defined through

$$\frac{1}{2}(r_+ - r_-) = \sqrt{\mu^2 - a^2} \equiv \epsilon\lambda, \quad (2.7)$$

is kept fixed as $\lambda \rightarrow 0$. The radius ℓ given in (2.6) is also fixed. Using

$$\begin{aligned} \frac{dr^2}{\Delta} &= \frac{dU^2}{U^2 - \epsilon^2}, \\ \Sigma &= \frac{\ell^2}{2}(1 + \cos^2 \theta) + \mathcal{O}(\lambda), \\ (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta &= \ell^4 \left[1 + \frac{\sqrt{8}}{\ell} \lambda U + \mathcal{O}(\lambda^2) \right], \end{aligned} \quad (2.8)$$

the resulting line element reads

$$ds^2 = \frac{1 + \cos^2 \theta}{2} \left[-\frac{U^2 - \epsilon^2}{\ell^2} dt^2 + \frac{\ell^2}{U^2 - \epsilon^2} dU^2 + \ell^2 d\theta^2 \right] + \ell^2 \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} \left(d\phi + \frac{U}{\ell^2} dt \right)^2. \quad (2.9)$$

This near extreme horizon geometry modifies the original NHEK geometry (2.5) only in the AdS_2 component. In fact, the (t, U) term in the square brackets remains locally AdS_2 with radius of curvature ℓ but the global structure is modified into a black hole geometry with horizon located at $U = \epsilon$. We interpret this deformation as an excitation above the extreme Kerr. The near extreme limit described here is the same considered in [29, 30] (and a Kerr analogue of “Limit 2” in [31]).

2.2 Phenomenology of Kerr thermodynamics

It is instructive to consider Kerr thermodynamics in the near extreme limit.

When manipulating thermodynamic formulae it is useful to introduce the Planck mass M_P and Planck length l_P through

$$G_4 = M_P^{-2} = l_P^2. \quad (2.10)$$

In this notation, we are interested in near extremal Kerr black holes for which

$$M = \sqrt{J} M_P + E, \quad (2.11)$$

with the excitation energy $E \ll \sqrt{J} M_P$. In this regime an important scale is the AdS_2 scale (2.6)

$$\ell = \frac{1}{\sqrt{2}}(r_+ + r_-) = \sqrt{2} M l_P^2 = \sqrt{2J} l_P + \dots \quad (2.12)$$

We can recast the precise expansion parameter here and in similar expressions below as $E\ell/J \ll 1$.

The black hole entropy for a general Kerr black hole is

$$S = \frac{A}{4G_4} = 2\pi \left(M^2 l_P^2 + \sqrt{M^4 l_P^4 - J^2} \right). \quad (2.13)$$

Expanding in the near extremal limit (2.11), the entropy becomes

$$S = 2\pi \left(J + \sqrt{4J^{3/2}El_P} \right) + \mathcal{O}(E\ell) . \tag{2.14}$$

The expression takes the suggestive form

$$S = 2\pi \left(\frac{c_L}{12} + \sqrt{\frac{c_R h_R}{6}} \right) + \dots , \tag{2.15}$$

with central charges

$$c_L = c_R = 12J , \tag{2.16}$$

and weight

$$h_R = 2E\sqrt{J}l_P . \tag{2.17}$$

In summary: in the extremal case $E = 0$, the entropy is a ground state entropy, corresponding to Cardy’s formula with $h_L = \frac{c}{24}$. On the other hand, the departure from the extremal entropy is given by a more conventional Cardy formula.

In our decoupling limit we parametrize the excitation energy by the nonextremality parameter ϵ (introduced in (2.7)). It is related to the excitation energy E (introduced in (2.11)) through

$$\epsilon\lambda = \sqrt{\mu^2 - a^2} = \frac{1}{M} \sqrt{M^2 l_P^4 - J^2} = \frac{1}{\sqrt{JM_P^2}} \sqrt{4J^{3/2}El_P} = \sqrt{4\sqrt{J}El_P^3} . \tag{2.18}$$

The weight (2.17) assigned to the excitation energy becomes

$$h_R = 2E\sqrt{J}l_P = \frac{\epsilon^2 \lambda^2}{2l_P^2} . \tag{2.19}$$

The expected level is small because typical (comoving) energies in the near horizon are small compared to typical asymptotic energies. According to (2.4) we have $\partial_t = \lambda\partial_{t'}$. However, since the level (2.19) is of order $\mathcal{O}(\lambda^2)$, the excitation energy will be parametrically small even compared to typical energies in AdS₂. The “non-relativistic” form (2.19) of the energy is wellknown from lightcone physics. In the present context the relevant excitations move at nearly the speed of light due to the rapid rotation of the black hole.

In order to “understand” the near extreme Kerr entropy (2.15) we must account for the two central charges (2.16) and the excitation level (2.19). In the next four sections we develop relevant results before returning to Kerr thermodynamics in section 7.

3 The 2D perspective

Our approach to Kerr dynamics focusses on the AdS₂ component of the near horizon geometry (2.9). In this section we determine the 2D effective theory in which this geometry appears as a solution.

3.1 The effective 2D theory

The 2D theory contains a general 2D metric

$$ds_2^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (3.1)$$

with $\mu, \nu = 1, 2$, a gauge connection encoding the rotation

$$\mathcal{B} = \mathcal{B}_\mu(x) dx^\mu, \quad (3.2)$$

and one scalar field ψ that couples to the size of the two angular coordinates. In other words, we consider those 4D field configurations that take the form

$$ds_4^2 = \frac{1}{2}(1 + \cos^2 \theta) \left[ds_2^2 + e^{-2\psi} \ell^2 d\theta^2 \right] + e^{-2\psi} \ell^2 \left(\frac{2 \sin^2 \theta}{1 + \cos^2 \theta} \right) [d\phi + \mathcal{B}]^2. \quad (3.3)$$

In the following we raise and lower all indices using $g_{\mu\nu}$ and the covariant derivatives ∇_μ are also formed using $g_{\mu\nu}$. The gauge field strength is denoted $\mathcal{G} = d\mathcal{B}$.

The 4D action is simply the Einstein Hilbert action

$$S_4 = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g_4} R^{(4)}. \quad (3.4)$$

The class of 4D metrics we consider (3.3) have Ricci scalar

$$R^{(4)} = \frac{2}{1 + \cos^2 \theta} \left[R^{(2)} + e^{2\psi} \frac{2}{\ell^2} - 2 e^{2\psi} \nabla^2 e^{-2\psi} + \frac{1}{2} e^{4\psi} \nabla_\mu e^{-2\psi} \nabla^\mu e^{-2\psi} \right] - 2 (1 + \cos^2 \theta)^{-3} \sin^2 \theta \left[e^{-2\psi} \ell^2 \mathcal{G}^2 + e^{2\psi} \frac{2}{\ell^2} \right], \quad (3.5)$$

and determinant

$$\sqrt{-g_4} = \frac{\ell^2}{2} e^{-2\psi} \sin \theta (1 + \cos^2 \theta) \sqrt{-g}. \quad (3.6)$$

Inserting these expressions into the 4D action (3.4) and integrating by parts using

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta (1 + \cos^2 \theta)^{-2} \sin^3 \theta = 2\pi, \quad (3.7)$$

we find the 2D effective action

$$S_{\text{Kerr}} = \frac{\ell^2}{4G_4} \int d^2x \sqrt{-g} \left[e^{-2\psi} R^{(2)} + \frac{1}{\ell^2} + 2 \nabla_\mu e^{-\psi} \nabla^\mu e^{-\psi} - \frac{\ell^2}{2} e^{-4\psi} \mathcal{G}^2 \right]. \quad (3.8)$$

We want to analyze this action. It is almost identical to the action considered in [12], but the dependence on the dilaton is different.

3.2 Solutions to the 2D theory

To get started, we determine some of the solutions to the 2D effective theory. For constant ψ , the equations of motion are

$$\begin{aligned} R^{(2)} - e^{-2\psi} \ell^2 \mathcal{G}^2 &= 0, \\ \frac{1}{2} \left(e^{4\psi} \frac{1}{\ell^2} - \frac{\ell^2}{2} \mathcal{G}^2 \right) g_{\mu\nu} + \ell^2 \mathcal{G}_{\mu\rho} \mathcal{G}_\nu{}^\rho &= 0, \\ \nabla_\mu \mathcal{G}^{\mu\nu} &= 0. \end{aligned} \tag{3.9}$$

Contracting the second equation in (3.9) we get

$$\mathcal{G}^2 = -e^{4\psi} \frac{2}{\ell^4}, \tag{3.10}$$

and the curvature is given by

$$R^{(2)} = -e^{2\psi} \frac{2}{\ell^2}. \tag{3.11}$$

The solutions with constant ψ are therefore locally AdS₂ with radius

$$\ell_{\text{AdS}} = e^{-\psi} \ell. \tag{3.12}$$

We will loosely refer to ℓ as the AdS₂ radius, since we can take $\psi = 0$ after solving the corresponding equation of motion.

The near horizon Kerr geometry (2.9) should be a solution to this theory. Indeed, we can verify that this is the case with

$$\begin{aligned} e^{-2\psi} &= 1, \\ \ell^2 &= 2G_4 J. \end{aligned} \tag{3.13}$$

Also, inserting the equations of motion (3.10), (3.11) in the 2D action (3.8), we find that the on-shell bulk action vanishes, as expected since $R^{(4)} = 0$ for the Kerr solution.

It is instructive to consider more general solutions. Without loss of generality we can choose a gauge where the metric takes the form

$$ds^2 = e^{-2\psi} d\rho^2 + \gamma_{tt} dt^2, \tag{3.14}$$

and the gauge field satisfies $\mathcal{B}_\rho = 0$. The solutions to the equations of motion in this case are²

$$\begin{aligned} \gamma_{tt} &= -\frac{1}{4} e^{-2\psi} \left(e^{\rho/\ell} - f(t) e^{-\rho/\ell} \right)^2, \\ \mathcal{B}_t &= \frac{1}{2\ell} e^{\rho/\ell} \left(1 - \sqrt{f(t)} e^{-\rho/\ell} \right)^2. \end{aligned} \tag{3.15}$$

The ρ -independent term in \mathcal{B}_t is arbitrary, a residual gauge function. We chose that term so that \mathcal{B}_t becomes a complete square, because this is the correct gauge for applications to

²The notation is related to that of [12] through $f(t) = -h_1(t)$, $\ell = L/2$, $\gamma_{tt} = h_{tt}$.

black holes [32]. The function $f(t)$ is arbitrary (except that $f(t) \geq 0$ in the gauge we have presented the solution in).

For any function $f(t)$, the asymptotic behavior

$$\begin{aligned}\gamma_{tt}^{(0)} &= -\frac{1}{4}e^{-2\psi^{(0)}}e^{2\rho/\ell}, \\ \mathcal{B}_t^{(0)} &= \frac{1}{2\ell}e^{\rho/\ell}, \\ \psi^{(0)} &= \text{constant},\end{aligned}\tag{3.16}$$

defines asymptotically AdS₂ configurations. We do not impose boundary conditions on the other AdS₂ boundary. It may be an issue that the solutions (3.16) are non-normalizable there [28]. On the other hand, in the Euclidean theory it is certainly consistent to focus on solutions with one boundary.³

The effective 2D geometries appearing in extremal Kerr (2.5) and near-extremal Kerr (2.9) are recovered through the coordinate transformation

$$U = \frac{\ell}{2}e^{\rho/\ell} \left(1 + \frac{\epsilon^2}{\ell^2}e^{-2\rho/\ell} \right).\tag{3.17}$$

The value $f(t) = -1$ corresponds to global AdS₂, the value $f(t) = 0$ corresponds to the extremal Kerr solution (2.5), and

$$f(t) = \frac{\epsilon^2}{\ell^2},\tag{3.18}$$

are the near extremal Kerr solutions (2.9). The terminology mimics the corresponding situation for BTZ solutions in AdS₃.

For later calculations, we record the extrinsic curvature at the boundary,

$$K = \frac{1}{2}\gamma^{tt}n^\mu\partial_\mu\gamma_{tt} = \frac{1}{\ell}e^\psi,\tag{3.19}$$

where the normal vector has component $n^\rho = \sqrt{g^{\rho\rho}} = e^\psi$.

4 The renormalized boundary action

In this section we determine the renormalized boundary action of the AdS₂ space. Our treatment follows the standard procedure for AdS/CFT correspondence in any dimension [33, 34], with the adaptation to AdS₂ developed in [12, 16, 32].

4.1 Boundary counterterms

We first need to determine the boundary terms

$$S_{\text{bdy}} = S_{\text{GH}} + S_{\text{counter}},\tag{4.1}$$

that give a well-defined variational principle. The Gibbons-Hawking term is given by

$$S_{\text{GH}} = \frac{\ell^2}{2G_4} \int dt \sqrt{-\gamma} e^{-2\psi} K.\tag{4.2}$$

³We thank A. Sen for this argument.

The counterterms take the local form

$$S_{\text{counter}} = \frac{\ell^2}{2G_4} \int dt \sqrt{-\gamma} \left[\lambda e^{-\psi} + m e^{-3\psi} \mathcal{B}_a \mathcal{B}^a \right], \quad (4.3)$$

with λ and m constants to be determined. The power of $e^{-\psi}$ in each term in (4.3) is determined by the dependence on ψ of the metric and gauge field. Using the expressions from the previous section we have $K \sim e^\psi$ and $\mathcal{B}_a \mathcal{B}^a \sim e^{2\psi}$, then each term in (4.2) and (4.3) scale as $e^{-2\psi}$ and therefore the constants λ and m will only depend on the scale ℓ . The consistency of the second term in (4.3) with gauge invariance was established in [12] for any gauge function that remains finite at infinity.

The variation of the full action, including boundary terms, takes the form

$$\delta S = \int dt \sqrt{-\gamma} \left[\pi^{ab} \delta \gamma_{ab} + \pi_\psi \delta \psi + \pi^a \delta \mathcal{B}_a \right] + \text{bulk terms}, \quad (4.4)$$

where the π are momenta that receive contributions from the variations of both the bulk and the boundary action. By explicit computation we find

$$\begin{aligned} \pi^{tt} &= \frac{\ell^2}{4G_4} \left(\lambda e^{-\psi} \gamma^{tt} + m e^{-3\psi} \gamma^{tt} \mathcal{B}_a \mathcal{B}^a - 2m e^{-3\psi} \mathcal{B}^t \mathcal{B}^t \right), \\ \pi^t &= \frac{\ell^2}{4G_4} \left(-2e^{-4\psi} \ell^2 n_\mu \mathcal{G}^{\mu t} + 4m e^{-3\psi} \mathcal{B}^t \right), \\ \pi_\psi &= -\frac{\ell^2}{2G_4} \left(2e^{-2\psi} K + \lambda e^{-\psi} + 3m e^{-3\psi} \mathcal{B}_a \mathcal{B}^a \right). \end{aligned} \quad (4.5)$$

Using (3.19) for the extrinsic curvature and (3.16) for the asymptotic geometry we find

$$\begin{aligned} \pi^{tt} &= \frac{\ell^2}{4G_4} \left(\lambda + \frac{m}{\ell^2} \right) e^{-\psi^{(0)}} \gamma_{(0)}^{tt}, \\ \pi^t &= \frac{\ell^2}{4G_4} \left(-1 + 2\frac{m}{\ell} \right) e^{-3\psi^{(0)}} \gamma_{(0)}^{tt} e^{\rho/\ell}, \\ \pi_\psi &= -\frac{\ell^2}{2G_4} \left(2\frac{1}{\ell} + \lambda - 3\frac{m}{\ell^2} \right) e^{-\psi^{(0)}}. \end{aligned} \quad (4.6)$$

The vanishing of these three boundary momenta is ensured by the two conditions

$$m = \frac{\ell}{2}, \quad \lambda = -\frac{1}{2\ell}. \quad (4.7)$$

Collecting results, we have determined the full action including boundary counterterms

$$\begin{aligned} S &= \frac{\ell^2}{4G_4} \int d^2x \sqrt{-g} \left[e^{-2\psi} R^{(2)} + \frac{1}{\ell^2} + 2\nabla_\mu e^{-\psi} \nabla^\mu e^{-\psi} - e^{-4\psi} \frac{\ell^2}{2} \mathcal{G}^2 \right] \\ &+ \frac{\ell^2}{2G_4} \int dt \sqrt{-\gamma} \left[e^{-2\psi} K - \frac{1}{2\ell} e^{-\psi} + \frac{\ell}{2} e^{-3\psi} \mathcal{B}_a \mathcal{B}^a \right]. \end{aligned} \quad (4.8)$$

4.2 Boundary currents

As we will see in the following, there are special difficulties in two dimensions that we have not yet addressed. Ignoring these for a moment, we proceed in the standard manner and compute the response functions on the boundary

$$\begin{aligned} T^{ab} &= \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ab}}, \\ J^a &= \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \mathcal{B}_a}. \end{aligned} \tag{4.9}$$

For the action (4.8), we find

$$\begin{aligned} T_{tt} &= -\frac{\ell^2}{4G_4} \left(\frac{1}{\ell} e^{-\psi} \gamma_{tt} + \ell e^{-3\psi} \mathcal{B}_t \mathcal{B}_t \right), \\ J_t &= \frac{\ell^2}{2G_4} e^{-3\psi} \left(-e^{-\psi} \ell^2 n^\mu \mathcal{G}_{\mu t} + \ell \mathcal{B}_t \right). \end{aligned} \tag{4.10}$$

As an example, we evaluate these expressions for our AdS₂ black hole solution. For easy reference, we write the metric and the gauge field for the black hole which were previously given by (3.15) with $f(t)$ the constant value (3.18),

$$\begin{aligned} \gamma_{tt} &= -\frac{1}{4} e^{-2\psi} e^{2\rho/\ell} \left(1 - \frac{\epsilon^2}{\ell^2} e^{-2\rho/\ell} \right)^2, \\ \mathcal{B}_t &= \frac{1}{2\ell} e^{\rho/\ell} \left(1 - \frac{\epsilon}{\ell} e^{-\rho/\ell} \right)^2. \end{aligned} \tag{4.11}$$

As we noted earlier, the constant term in \mathcal{B}_t is fixed in the black hole context, by the condition that the corresponding Euclidean solution is regular at the horizon. Now, inserting (4.11) in (4.10) we find

$$\begin{aligned} T_{tt} &= \frac{\ell}{4G_4} e^{-3\psi} \left(\frac{\epsilon}{\ell} e^{\rho/\ell} - \frac{2\epsilon^2}{\ell^2} + \dots \right), \\ J_t &= \frac{\ell^2}{2G_4} e^{-3\psi} \left(-\frac{\epsilon}{\ell} + \frac{\epsilon^2}{\ell^2} e^{-\rho/\ell} \right). \end{aligned} \tag{4.12}$$

In each expression the leading divergence cancelled. Indeed, we designed the boundary counterterms to ensure this property. However, the remaining expression for the energy momentum tensor still diverges as $\rho \rightarrow \infty$. This behavior is special to two spacetime dimensions, and it is entirely unacceptable.⁴ In the following sections we will carefully take into account some of the special features in two spacetime dimensions and find a more satisfactory result.

⁴It is worth noting the response functions (4.10) would have been finite, if we had fixed the residual gauge so that there is no ρ -dependent term in \mathcal{B}_t . The divergence in the energy momentum tensor can thus be removed at the expense of having a singular Euclidean gauge field at the horizon. This possibility may be relevant in some situations but we will not rely on that cancellation here.

5 Boundary conditions and asymptotic symmetries

In the previous section we treated diffeomorphisms and gauge transformations as independent. However, Hartman and Strominger noticed that only a combination of these transformations are consistent with the gauge conditions [13]. In this section we incorporate this feature.

We have assumed that the metric is written in Fefferman-Graham form (3.14), with $g_{\rho\rho}$ and $g_{\rho t}$ specified as a gauge conditions, along with the asymptotic behavior (3.16). The diffeomorphisms that preserve these conditions are those that satisfy

$$\delta_\epsilon g_{\rho\rho} = 0, \quad \delta_\epsilon g_{\rho t} = 0, \quad \delta_\epsilon g_{tt} = 0 \cdot e^{2\rho/\ell} + \dots \quad (5.1)$$

General diffeomorphisms transforms the metric as

$$\begin{aligned} \delta_\epsilon g_{\mu\nu} &= \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \\ &= g_{\nu\lambda} \partial_\mu \epsilon^\lambda + g_{\mu\lambda} \partial_\nu \epsilon^\lambda + \epsilon^\lambda \partial_\lambda g_{\mu\nu} . \end{aligned} \quad (5.2)$$

Imposing the conditions (5.1) we find that the allowed coordinate transformations are

$$\epsilon^\rho = -\ell \partial_t \xi(t), \quad \epsilon^t = \xi(t) + 2\ell^2 \left(e^{2\rho/\ell} - f(t) \right)^{-1} \partial_t^2 \xi(t), \quad (5.3)$$

where $\xi(t)$ is an arbitrary function. We used the explicit solutions (3.15). Under these allowed diffeomorphisms the boundary metric transforms as

$$\delta_\epsilon \gamma_{tt} = -e^{-2\psi} (1 - f(t) e^{-2\rho/\ell}) \left[-f(t) \partial_t \xi - \frac{1}{2} \partial_t f(t) \xi + \ell^2 \partial_t^3 \xi \right] . \quad (5.4)$$

The next step is to consider the transformation of the gauge field \mathcal{B}_μ under the allowed diffeomorphisms. A vector transforms according to

$$\begin{aligned} \delta_\epsilon \mathcal{B}_\mu &= \epsilon^\lambda \nabla_\lambda \mathcal{B}_\mu + \mathcal{B}_\lambda \nabla_\mu \epsilon^\lambda \\ &= \epsilon^\lambda \partial_\lambda \mathcal{B}_\mu + \mathcal{B}_\lambda \partial_\mu \epsilon^\lambda , \end{aligned} \quad (5.5)$$

under a general diffeomorphism. Considering just the allowed diffeomorphisms (5.3), the transformation of the radial component becomes

$$\delta_\epsilon \mathcal{B}_\rho = -2 \left(1 + \sqrt{f(t)} e^{-\rho/\ell} \right)^{-2} e^{-\rho/\ell} \partial_t^2 \xi . \quad (5.6)$$

This clearly violates the gauge condition $\mathcal{B}_\rho = 0$.

To restore the gauge condition we compensate by a gauge transformation with gauge function

$$\Lambda = -2\ell e^{-\rho/\ell} \left(1 + \sqrt{f(t)} e^{-\rho/\ell} \right)^{-1} \partial_t^2 \xi , \quad (5.7)$$

chosen so that the combined diffeomorphism and gauge transformation gives

$$\delta_{\epsilon+\Lambda} \mathcal{B}_\rho = \delta_\epsilon \mathcal{B}_\rho + \partial_\rho \Lambda = 0 . \quad (5.8)$$

It is the combination of allowed diffeomorphisms (5.3) with the compensating gauge transformation (5.7) that generate legitimate symmetries of the theory [13]. The variation of the time component of the gauge field under the symmetry is

$$\delta_{\epsilon+\Lambda}\mathcal{B}_t = -\frac{e^{-\rho/\ell}}{\ell} \left[-f(t)\partial_t\xi - \frac{1}{2}\partial_t f(t)\xi + \ell^2\partial_t^3\xi \right] - \frac{1}{\ell}\partial_t(\sqrt{f}\xi) . \quad (5.9)$$

The transformation of the stress tensor (4.10) under the combinations of diffeomorphisms and U(1) transformations that constitute a symmetry

$$\delta_{\epsilon+\Lambda}T_{tt} = T_{tt}\partial_t\xi + \xi\partial_t T_{tt} + \frac{\ell^3}{2G_4}e^{-3\psi}\partial_t^3\xi + \mathcal{O}(e^{-\rho/\ell}), \quad (5.10)$$

and the transformation of the U(1) current (4.10) under the same transformations is

$$\delta_{\epsilon+\Lambda}J_t = J_t\partial_t\xi + \xi\partial_t J_t - \frac{\ell^4}{G_4}e^{-3\psi}e^{-\rho/\ell}\partial_t^3\xi + \mathcal{O}(e^{-2\rho/\ell}) . \quad (5.11)$$

The transformation laws are very interesting because they almost resemble the standard ones in CFT. In fact, if we introduce the central charge as

$$\delta_{\epsilon+\Lambda}T_{tt} = T_{tt}\partial_t\xi + \xi\partial_t T_{tt} - \frac{c}{12}\ell_{\text{AdS}}\partial_t^3\xi , \quad (5.12)$$

where the AdS radius was defined in (3.12), gives

$$c = \frac{6}{G_4}\ell_{\text{AdS}}^2 , \quad (5.13)$$

up to a sign. The AdS₂ radius for near extreme Kerr (3.13) then gives

$$c_{\text{Kerr}} = 12J . \quad (5.14)$$

This is exactly the value that was found in Kerr/CFT [1] using entirely different methods. Despite this apparent success, essential concerns remain. Apart from the unacceptable sign, the stress-tensor T_{tt} is not related in any obvious way to the combination of diffeomorphism and gauge transformation that is an actual symmetry of the system and it does not have the correct conformal weight. In the next section we develop a more systematic point of view that will determine the central charge unambiguously.

6 Conserved charges

In this section we construct the conserved currents that generate diffeomorphisms and gauge transformations. We also compute the central charge and U(1) level associated to these generators.

6.1 Noether charge for time translation

Under a diffeomorphism $x^\mu \rightarrow x^\mu + \epsilon^\mu$ the variation of the action is

$$\delta_\epsilon S = \frac{1}{2} \int dt \sqrt{-\gamma} T^{ab} \delta_\epsilon \gamma_{ab} + \int dt \sqrt{-\gamma} J^a \delta_\epsilon \mathcal{B}_a + (\text{e.o.m}), \quad (6.1)$$

where

$$\begin{aligned} \delta_\epsilon \gamma_{ab} &= \nabla_a \epsilon_b + \nabla_b \epsilon_a, \\ \delta_\epsilon \mathcal{B}_a &= \epsilon^b \nabla_b \mathcal{B}_a + \mathcal{B}_b \nabla_a \epsilon^b. \end{aligned} \quad (6.2)$$

To define the conserved charge associated to ϵ^a , consider a regulator function F such that in the neighborhood of the boundary we have $F = 1$. Then define the variation of a field Φ as

$$\delta_{\epsilon, F} \Phi \equiv (\delta_{F\epsilon} - F\delta_\epsilon) \Phi. \quad (6.3)$$

(For further details, see e.g. [35]). Implementing this variation for the action we get

$$\begin{aligned} \delta_{\epsilon, F} S &= \int dt \sqrt{-\gamma} T^{ab} \epsilon_a \nabla_b F + \int dt \sqrt{-\gamma} J^a \mathcal{B}_a \epsilon^b \nabla_b F \\ &= \sqrt{-\gamma} \gamma^{tt} (T_{tt} + J_t \mathcal{B}_t) \epsilon^t - \int dt \nabla_b (T^{ba} \epsilon_a + J^b \mathcal{B}_a \epsilon^a) F, \end{aligned} \quad (6.4)$$

where in the second line we integrated by parts and used $F = 1$ near the boundary. The last term vanishes due to Noether's theorem so the charge that generates infinitesimal temporal diffeomorphisms ϵ^t becomes

$$Q_\epsilon = \sqrt{-\gamma} \gamma^{tt} (T_{tt} + J_t \mathcal{B}_t). \quad (6.5)$$

It is quite unfamiliar to find that the generator of translations involves a matter term, along with the energy momentum tensor. The matter term does in fact appear in other contexts as well, but it is usually negligible because matter fields decay more rapidly than the metric field. The situation here is special since $\mathcal{B}_t \sim e^{\rho/\ell}$ increases rapidly as the boundary is approached.

To make things explicit, we consider the black hole solution (4.11), for which the expressions (4.12) give

$$T_{tt} + \mathcal{B}_t J_t = \frac{\ell}{4G_4} e^{-3\psi} \frac{\epsilon^2}{\ell^2}. \quad (6.6)$$

Recall that we previously found a divergence in the energy momentum tensor (4.12). However, the additional matter term contains a divergence of its own, such that the sum is finite.

The metric factors in (6.5) redshift Q_ϵ to zero in the limit $\rho \rightarrow \infty$. In this sense we will still assign vanishing energy to all excitations, as expected from general principles in two dimensions. (Some recent discussions are [28–30]). We will see in section 7.2 that we nevertheless have a meaningful concept of energy.

6.2 The U(1) charge

The electric charge is given by the variation of the action with respect to the field strength \mathcal{G} . We can implement this prescription by introducing an off-shell electric potential ϕ_e as an overall proportionality constant

$$\mathcal{G}_{\rho t} = \phi_e \cdot (\mathcal{G}_{\rho t})_0, \quad \mathcal{B}_t = \phi_e \cdot (\mathcal{B}_t)_0. \quad (6.7)$$

The subindex “0” denotes the on-shell values we have used hitherto, corresponding to $\phi_e = 1$. The U(1) charge that generates gauge transformations is the variation of the on-shell action

$$Q_{\text{U}(1)} = \left. \frac{\delta S}{\delta \phi_e} \right|_{\phi_e=1} = \int dt \sqrt{-\gamma} J^a \left. \frac{\delta \mathcal{B}_a}{\delta \phi_e} \right|_{\phi_e=1} = \int dt \sqrt{-\gamma} \gamma^{tt} J_t \mathcal{B}_t. \quad (6.8)$$

We used the definition of the current (4.9).

To make the formula explicit we insert the asymptotic behavior (3.15) and find

$$Q_{\text{U}(1)} = -\frac{1}{\ell} \int dt e^\psi J_t, \quad (6.9)$$

to the leading order. This is a very conventional expression for the charge, essentially the time component of the current.

As we have already remarked, the generator of diffeomorphisms (6.5) will vanish, due to the overall metric factors. Therefore the path integral will be dominated by the gauge theory generator (6.8). This is familiar from Sen’s quantum entropy function. [16, 32].

6.3 The generator of gauge transformations

We already showed that the generator of diffeomorphisms differs from the the energy momentum tensor, due to large gauge fields near the boundary. Next we will show that the generator of gauge transformations differs from the U(1) charge, for the same reason.

The variation of the action (5.7) under a gauge transformation $\delta \mathcal{B}_a = \partial_a \Lambda$ is

$$\delta_\Lambda S = \int dt \sqrt{-\gamma} \mathcal{J}^a \partial_a \Lambda, \quad (6.10)$$

with

$$\mathcal{J}_t = \frac{\ell^3}{2G_4} e^{-3\psi} \mathcal{B}_t. \quad (6.11)$$

To be more precise we proceed as for diffeomorphisms: introduce a regulator function F and define

$$\delta_{\Lambda, F} \Phi \equiv (\delta_{F\Lambda} - F\delta_\Lambda) \Phi. \quad (6.12)$$

The variation of the action is then

$$\begin{aligned} \delta_{F, \Lambda} S &= \int dt \sqrt{-\gamma} \mathcal{J}^a \Lambda \partial_a F \\ &= \sqrt{-\gamma} \mathcal{J}^t \Lambda - \int dt \partial_a (\sqrt{-\gamma} \mathcal{J}^a \Lambda) F. \end{aligned} \quad (6.13)$$

The last term in (6.13) vanishes due to Noether theorem, and hence we identify the generator of gauge transformations as

$$Q_\Lambda = \sqrt{-\gamma} \mathcal{J}^t = \frac{\ell^3}{2G_4} e^{-3\psi} \sqrt{-\gamma} \gamma^{tt} \mathcal{B}_t . \quad (6.14)$$

For example, we may evaluate this Noether charge on the black hole solution (4.11) and find

$$Q_\Lambda = -\frac{\ell^2}{2G_4} e^{-2\psi} \left(1 - 2\frac{\epsilon}{\ell} e^{-\rho/\ell} + \dots \right) . \quad (6.15)$$

In the NHEK geometry (3.13) the Noether charge takes the satisfying value

$$Q_\Lambda = -J . \quad (6.16)$$

The sign is just a matter of conventions.

6.4 Central charge

Now that we have identified the generators of diffeomorphism (6.5) and gauge transformation (6.14), we can discuss if there is a central charge associated to the combined asymptotic symmetries constructed in section 5.

The appropriate charge associated to the combined transformation ϵ^μ (5.3) and Λ (5.7) is a linear combination of Q_ϵ and Q_Λ . To properly add these two terms, we consider the asymptotic behavior of the transformation parameters

$$\begin{aligned} \epsilon^t &= \xi(t) + 2\ell^2 e^{-2\rho/\ell} \partial_t^2 \xi + \dots , \\ \Lambda &= -2\ell e^{-\rho/\ell} \partial_t^2 \xi + \dots . \end{aligned} \quad (6.17)$$

To leading order we can write the gauge parameter as a function of the diffeomorphisms

$$\Lambda = \ell \mathcal{B}_a \partial_\rho \epsilon^a + \dots . \quad (6.18)$$

Therefore, when we considered the combined generator

$$Q_{(\epsilon+\Lambda)}(\epsilon + \Lambda) = Q_\epsilon \epsilon + Q_\Lambda \Lambda , \quad (6.19)$$

we will have a contribution from Q_ϵ given by (6.5) and, from the gauge generator, a term of the form

$$Q_\Lambda \Lambda = -\sqrt{-\gamma} \gamma^{tt} \mathcal{B}_t \mathcal{J}_t \epsilon^t + \dots . \quad (6.20)$$

where we used $Q_\Lambda \mathcal{B}_t \sim e^{\rho/\ell}$ to integrate by parts the ρ derivative in (6.18).

This last expression has to be taken with some caution. From sections 6.1 and 6.3, we have $Q_\Lambda \Lambda$ is of the same order as $Q_\epsilon \epsilon$, hence (6.20) is relevant to the transformation of (6.19). But if we decompose (6.17) in Fourier modes, i.e. $\xi_n \sim e^{int}$, the gauge function will only affect higher order modes (Q_Λ will not contribute to the zero mode associated to the energy eigenvalue). When evaluating (6.20) the transformation parameter ϵ^t has to be taken of order $e^{-2\rho/\ell}$.

Lets first focus on the diffeomorphisms. From (6.5) the relevant combination is $T_{tt} + \mathcal{B}_t J_t$. Using (5.9) - (5.11) we find

$$\delta_{\epsilon+\Lambda} (T_{tt} + \mathcal{B}_t J_t) = 2 (T_{tt} + \mathcal{B}_t J_t) \partial_t \xi + \xi \partial_t (T_{tt} + \mathcal{B}_t J_t) + \mathcal{O}(\sqrt{f} e^{-\rho/\ell}) \quad (6.21)$$

The generator transforms as a tensor of weight two as expected for a stress tensor of the CFT. The constant term in (5.10) does contribute a central term, but it is cancelled by the contribution from gauge field. All the sub-leading corrections are proportional to the fluctuations $\sqrt{f} = \epsilon/\ell$ and redshifted, so asymptotically we have a vanishing central term for Q_ϵ .

The gauge contribution to the transformation of (6.19) reduces to

$$\delta_{\epsilon+\Lambda} (\mathcal{B}_t \mathcal{J}_t) = \xi \partial_t (\mathcal{B}_t \mathcal{J}_t) - \frac{\ell^3}{2G_4} e^{-3\psi} \partial_t^3 \xi \quad (6.22)$$

As written, the weight of the tensor vanishes (corresponding to asymptotic behavior $e^{-2\rho/\ell}$) instead being two (corresponding to asymptotically constant). This is an artifact of the two scales in the problem. As we already mentioned we are interested in $\epsilon^t \sim e^{-2\rho/\ell}$ and for such excitations the dimension is effectively two.

Using the same normalization of central charge as in (5.12), we identify the central charge as

$$c = \frac{6}{G_4} \ell_{AdS}^2 . \quad (6.23)$$

This is the central charge associated to the charge $Q_{\epsilon+\Lambda}$. The diffeomorphism reflect the conformal structure of the generator, and after combined with the gauge transformation we obtain a central term. It is also exactly what we expected from the discussion around equations (5.12) - (5.14), but here we are able to compute the central charge from a consistent generator.

6.5 Level

Finally, we consider the gauge transformation of the U(1) current and the associated level. The level k is defined as

$$\delta_\Lambda J_t = \frac{k}{2} \ell_{AdS} \partial_t \Lambda . \quad (6.24)$$

The gauge variation of the current (6.11) is

$$\delta_\Lambda J_t = e^{-2\psi} \frac{\ell^2}{2G_4} \left(e^{-\psi} \ell \right) \partial_t \Lambda , \quad (6.25)$$

so we find

$$k = \frac{1}{G_4} \ell_{AdS}^2 . \quad (6.26)$$

This value for the level is related to the central charge (6.23) in the simple manner

$$c = 6k , \quad (6.27)$$

as they were in our previous work [12].

The relation (6.27) is precisely the one expected between central charge and the level of the R-current in $N = 4$ supersymmetry. Since Kerr black holes break supersymmetry completely, there would *a priori* be no reason to encounter this relation here. However, in the context of the larger picture we outlined in the introduction (and elaborate in the discussion), this result is precisely what we expect. According to our interpretation the full theory carries (4,0) supersymmetry and it is the supersymmetric sector we probe here. That sector is in a *state* that breaks supersymmetry, because of the condensate of quanta carrying the Kerr angular momentum, but the *theory* should maintain supersymmetry. The relation (6.27) between central charge and current gives support to this picture.

7 Thermodynamics of near-extreme Kerr

In this section we work out the consequences of the results in the previous sections for the near extreme Kerr black hole.

7.1 The on-shell action

We define AdS₂ quantum gravity as a Euclidean path integral (even though we maintain Lorentzian notation), evaluated as function of boundary data. In particular it is a function of the inverse temperature and the Noether charge.

The Euclidean version of the black hole metric (4.11) has a conical singularity unless the Euclidean time coordinate has periodicity

$$\beta_0 = \frac{2\pi\ell}{\sqrt{f}} = \frac{2\pi\ell^2}{\epsilon} . \tag{7.1}$$

In asymptotically AdS₂ the physical inverse temperature becomes

$$T^{-1} = \beta = \frac{2\pi\ell^2}{\epsilon} \cdot \frac{1}{2} e^{-\psi} e^{\rho/\ell} \left(1 - \frac{\epsilon^2}{\ell^2} e^{-2\rho/\ell} \right) . \tag{7.2}$$

In boundary formulae like this one ρ is interpreted as a cut-off, as usual. In the limit $\rho \rightarrow \infty$ where the cutoff is removed, the physical temperature $T \rightarrow 0$, as expected for extremal black holes. We want to keep the finite temperature effects near extremality so we will keep the leading corrections, rather than taking the strict limit.

We evaluate the path integral in a sector where the Noether generator (6.15) takes a fixed value which we identify with the angular momentum or the central charge

$$Q_\Lambda = -\frac{\ell^2}{2G_4} e^{-2\psi} \left(1 - 2\frac{\epsilon}{\ell} e^{-\rho/\ell} + \dots \right) = -J = -\frac{c}{12} . \tag{7.3}$$

We can now compute the on-shell action from our renormalized boundary action (4.8). The bulk terms in the action (4.8) vanishes on the solution (4.11), as one can readily show

using the equations of motion (3.10), (3.11). The boundary terms give the on-shell action

$$\begin{aligned}
 I_M &= \frac{\ell^2}{2G_4} \int dt \sqrt{-\gamma} \left[e^{-2\psi} K - \frac{1}{2\ell} e^{-\psi} + \frac{\ell}{2} e^{-3\psi} \mathcal{B}_a \mathcal{B}^a \right] \\
 &= \frac{\ell^2}{2G_4} \cdot \beta \cdot \frac{2\epsilon}{\ell^2} e^{-\psi} e^{-\rho/\ell} \left[1 - \frac{\epsilon}{\ell} e^{-\rho/\ell} + \dots \right] \\
 &= \frac{\pi \ell^2}{G_4} e^{-2\psi} \left[1 - \frac{\epsilon}{\ell} e^{-\rho/\ell} + \dots \right] \\
 &= \frac{\pi c}{6} \left[1 + \frac{\epsilon}{\ell} e^{-\rho/\ell} + \dots \right] \\
 &= \frac{\pi c}{6} (1 + \pi T \ell_{\text{AdS}}) .
 \end{aligned} \tag{7.4}$$

We made the final expression more transparent by introducing the central charge using (7.3) and then the temperature (7.2).

The Euclidean action (which has the opposite sign of the Lorentzian action (7.4)) is essentially the free energy

$$\beta F = I_E = -I_M = -\frac{\pi c}{6} (1 + \pi T \ell_{\text{AdS}}) . \tag{7.5}$$

The entropy derived from the renormalized on-shell action then becomes

$$S = -\frac{\partial F}{\partial T} = \frac{\pi c}{6} + \frac{\pi^2 c}{3} T \ell_{\text{AdS}} . \tag{7.6}$$

The first term can be interpreted as the ground state entropy of the chiral half of a CFT with central charge $c_L = c$, as in the usual Kerr/CFT [1]. The second term is the Cardy expression for the other chiral half of the CFT with the same central charge $c_R = c$, excited to temperature T . The entropy (7.6) computed from the renormalized on-shell action is equivalent in form to the expression (2.15) we found directly from the standard Kerr entropy.⁵

7.2 The excitation energy

Our computation of the on-shell action assigns the energy

$$E = \frac{\partial(\beta F)}{\partial \beta} = \frac{\pi^2 c}{6} T^2 \ell_{\text{AdS}} , \tag{7.7}$$

to the excitations of the CFT. As a check on our computations we would like to recover this expression directly from the energy momentum tensor.

The energy is generated by time translations, i.e. diffeomorphisms with $\epsilon^t = 1$. In this case the compensating gauge transformations (5.7) vanish the energy can be computed directly from the (shifted) energy momentum tensor (6.6). We find the local energy

$$E_{\text{loc}} = T_{\hat{t}\hat{t}} + \mathcal{B}_{\hat{t}} J_{\hat{t}} = \frac{\epsilon^2}{\ell G_4} e^{-\psi} e^{-2\rho/\ell} = \frac{\pi^2 \ell^3}{G_4} T^2 e^{-3\psi} = \frac{\pi^2 c}{6} T^2 \ell_{\text{AdS}} . \tag{7.8}$$

⁵In (7.6) we write the entropy in a Cardy form by using our value of central charge and temperature, but the derivation of the entropy here does not use Cardy's formula. It would be interesting to investigate why (7.6) can be brought to this form. This requires a better understanding of the spectrum of the theory and modular invariance.

We used the boundary metric (3.16) to go to the local frame and then traded the cutoff for the temperature using (7.2). In the final step we used the central charge (6.23). Our final expression (7.8) computed from the local energy momentum tensor agrees with (7.7) computed from the on-shell action.

In our “phenomenological” analysis of the Kerr entropy, we determined the CFT level h_R as (2.19), using the macroscopic area law. In particular we found that the excitation energy should be of second order in the decoupling parameter λ , as expected for excitations described in the frame rotating at (nearly) the speed of light. In the CFT we similarly assign the excitations an energy (7.7) of second order in the small parameter T or, equivalently, or second order in the cut-off $e^{-\rho/\ell}$. The implied correspondence between parameters is

$$\lambda^2 = 2e^{-2\rho/\ell} . \tag{7.9}$$

It would be interesting to have an independent derivation of the relative numerical factor.

8 Discussion

There are by now several approaches to Kerr/CFT and, more generally, to AdS₂ quantum gravity. The subject faces many challenges and its entire consistency has been challenged. A strategy that may go far towards addressing the challenges that remain would be to relate the various approaches more precisely. In this spirit we comment on how our work is related to some other ideas:

The *Quantum Entropy Function* is a rather systematic approach to Euclidean AdS₂ quantum gravity [16, 32]. It is well established and it satisfies many consistency checks. Our approach reduces to the quantum entropy function in the extremal limit and then generalizes to where small excitations above extremality are included. To make this generalization explicit in the quantum entropy framework it must be shown that boundary conditions can be deformed consistently.

The *3D origin of AdS₂ Quantum Gravity* could help clarify many issues. The Euclidean version of this is quite well understood [27] but the Lorentzian theory raises some issues [28]. In Lorentzian signature AdS₂ can be embedded in to AdS₃ along a light cone direction. It then seems that our 2D theory can be interpreted as a slightly deformed embedding, along a direction that is near the light-cone but does not coincide with it. This would explain why we have access to both chiralities of the theory, corresponding to the direction along the light cone and transverse to it.

The *Asymptotic Symmetry Group of Kerr* involves the theory directly in 4D. We have reduced to 2D in a straightforward manner but the consistency was not established in detail. It would be interesting to extend our approach to the full NHEK geometry. That could help address some of the challenges facing the original Kerr/CFT [29, 30, 36], by providing a flexible Lagrangean framework. In addition this might provide some insight to boundary conditions and asymptotic symmetries studied in [10, 11, 37].

The justified focus on the immediate challenges to Kerr/CFT should not distract from the larger opportunities that seem to be appearing. The important point is not just whether or not the existence of 2D conformal symmetry and the applicability of Cardy’s formula

can be established on completely general grounds. The apparent ubiquity of Virasoro algebras is a surprise either way. In physical terms, many black holes apparently have an “effective string” structure in the spacetime theory, with excitations classified according to two chiral sectors of a CFT. It is far from obvious *a priori* that such a structure should apply to extremal Kerr, especially since the isometry group is just $SL(2, \mathbb{R}) \times U(1)$. It begs the question of whether more generic black holes, without any $SL(2, \mathbb{R})$ isometry at all, would similarly have two Virasoro algebras.

There is a hint of such a structure in the general entropy formula for Kerr

$$S = \frac{A}{4G_4} = 2\pi \left(M^2 l_P^2 + \sqrt{M^4 l_P^4 - J^2} \right), \tag{8.1}$$

which might be interpreted as a Cardy formula that applies all the way off extremality [38]. Indeed, the entropy of much more general 4D black holes (with many charges) takes a very similar form, with the angular momentum J appearing as a subtraction under the square root on the “supersymmetric” (R) side [19]. The two apparent levels satisfy a “level matching condition” [17]

$$N_L - N_R = (M^4 l_P^4) - (M^4 l_P^4 - J^2) = J^2, \tag{8.2}$$

that is integral also in much more general cases. We referred to this level matching condition already in the introduction. The general level matching condition can be expressed geometrically in terms of the outer and inner horizon areas as [18, 19]

$$\frac{1}{(8\pi G_4)^2} A_+ A_- = \text{integer} . \tag{8.3}$$

The origin of this type of relation is far from clear. It seems to indicate that the geometry “knows” about the division into two chiralities quite generally and that, in some way, the general theory is a combination of two factors, each of which has the structure familiar from BPS theory. This kind of prospect is perhaps the most compelling motivation for further development of the field.

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